4. S. E. Mikhailov and A. E. Osokin, "Construction of the fundamental solution for an anisotropic aging hereditary-type medium," Dokl. Akad. Nauk SSSR, 274, No. 2 (1984).
5. S. E. Mikhailov and A. E. Osokin, "Construction of the fundamental solutions for spatial and plane problems in the case of an anisotropic hereditary-elastic aging medium," Izv. Akad. Nauk SSSR, Mekh. Tverd. Tela, No. 5 (1986).
6. H. Bateman and A. Erdane, Higher Transcendental Functions (the Hypergeometric Function, the Legendre Function) [Russian translation], Fizmatgiz, Moscow (1973).
7. M. Abramovitz and M. Stigan (eds.), Handbook of Special Functions [Russian translation], Nauka, Moscow (1979).

CONTACT PROBLEM IN THE THEORY OF ELASTICITY FOR NARROW AREAS, WITH WEAR TAKEN INTO CONSIDERATION

A. N. Burmistrov

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1. We examine the spatial steady contact problem for the theory of elasticity in the presence of wear. Let a body 1 slide relative to body 2; let there be no wear in this case, and let the linear wear j for body 2 be proportional to the work of the force of friction [1]

$$j = K^*\mu l p_1,$$

where p_1 is the pressure; μ and K^* are the coefficients of friction and proportionality between the work of the force of friction and the volume of material removed; ℓ represents the friction path.

Let us choose an affine system of coordinates $Ox_1y_1z_1$, connected to the contact (the Oz_1 axis is perpendicular to the contact and directed toward body 1), so that $\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z$ exhibits unit length, and the angle between $\mathbf{e}_x$ and $\mathbf{e}_y$ is equal to β (see Fig. 1).

Let the field of the vector for the sliding velocity be uniformly plane-parallel: $\mathbf{V} = -v\mathbf{e}_y$, the area of contact $G_1 = \{(x_1, y_1):\ x_1^- \le x_1 \le x_1^+,\ y_1^-(x_1) \le y_1 \le y_1^+(x_1)\}$ [$y_1^\pm(x_1)$ are continuous functions]. The shape of the bodies and of the contact is independent of time. This hypothesis is valid, for example, in the following cases: a) 2 represents the half space; b) 1 is the rocking body and 2 is the bearing ring.

The equation from the theory of elasticity, with wear taken into consideration, has the form

$$\theta \iint\limits_{G_1} \frac{p_1(\xi_1, \eta_1)\, d\xi_1\, d\eta_1 \sin\beta}{r(\xi_1, \eta_1, x_1, y_1)} = w_1(x_1, y_1) - K^*\mu \int\limits_{y}^{y_1^+(x_1)} p_1(x_1, \eta_1)\, d\eta_1. \tag{1.1}$$

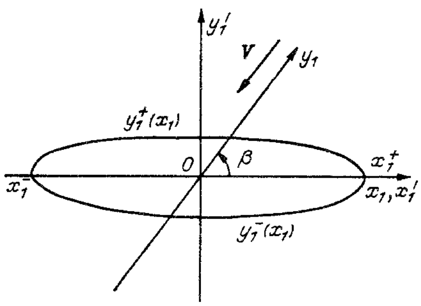

Fig. 1

Zhukovskii. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 4, pp. 68-76, July-August, 1990. Origdinal article submitted January 9, 1989.

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Here $\theta = \theta_1 + \theta_2$, $\theta_n = (1 - \nu_n^2)/(\pi E_n)$, ν_n and E_n are the Poisson coefficient and the modulus of body elasticity); w_1 represents the total elastic displacement; $r(\xi_1, \eta_1, x_1, y_1)$ is the distance between the points ξ_1, η_1 and x_1, y_1; n = 1 (2) corresponds to body 1 (2).

Let us assume that the characteristic dimension B of the contact along the Oy_1 axis is considerably smaller than the corresponding dimension of L along the Ox_1 axis. We will introduce the small parameter $\varepsilon = B/L$ and the dimensionless coordinates and variables: $x = x_1/L$, $\xi = \xi_1/L$, $y = y_1/B$, $\eta = \eta_1/B$, $x^{\pm} = x_1^{\pm}/L$, $y^{\pm} = y_1^{\pm}/B$, $\gamma = K^*\mu/(\theta \sin\beta)$, $p = \theta p_1 \sin\beta$, $w = w_1/B$. Equation (1.1) assumes the form

$$\iint\limits_G \frac{p(\xi, \eta)\, d\xi\, d\eta}{R_\varepsilon(\xi - x, \eta - y)} = w(x, y) - \gamma \int\limits_y^{y^+(x)} p(x, \eta)\, d\eta,$$
$$R_\varepsilon(u_1, u_2) = [u_1^2 + 2u_1u_2\varepsilon\cos\beta + \varepsilon^2 u_2^2]^{1/2}.$$

We will study this equation for the case of a narrow contact region ($\varepsilon \to 0$). An asymptotic equation was derived in [2] for an arbitrary curvilinear coordinate system as $\varepsilon \to 0$. In the affine coordinate system it is written in the form

$$\int\limits_{x^-}^{x^+} \frac{q(\xi) - q(x)}{|\xi - x|}\, d\xi + q(x) \ln\left[\frac{4(x^+ - x)(x - x^-)}{\varepsilon^2 \sin^2\beta}\right] = w(x, y) +$$
$$+ 2\int\limits_{y^-(x)}^{y^+(x)} p(x, \eta) \ln|y - \eta|\, d\eta - \gamma \int\limits_y^{y^+(x)} p(x, \eta)\, d\eta, \quad q(x) = \int\limits_{y^-(x)}^{y^+(x)} p(x, y)\, dy. \tag{1.2}$$

When $\gamma = 0$ Eq. (1.2), as demonstrated in [2], is brought to the form of a one-dimensional integral equation for q(x). Let us solve the analogous problem for $\gamma \neq 0$.

2. Having differentiated (1.2) with respect to y, we obtain

$$\int\limits_{y^-(x)}^{y^+(x)} \frac{p(x, \eta)}{\eta - y}\, d\eta - \frac{\gamma}{2} p(x, y) = \frac{1}{2} \frac{\partial w(x, y)}{\partial y}. \tag{2.1}$$

This singular integral equation for fixed x reduces to the Riemann boundary-value problem [3] for a function regular in a plane with a section $[y^-, y^+]$:

$$\Phi(z) = \frac{1}{2\pi i} \int\limits_{y^-(x)}^{y^+(x)} \frac{p(x, \eta)}{\eta - z}\, d\eta.$$

The boundary-value problem has the form

$$\Phi^+ = e^{2\pi\varphi i}\Phi^- - \frac{w_y}{\alpha} e^{\pi\varphi i}, \quad \Phi^+ - \Phi^- = p, \tag{2.2}$$

where $\varphi = \frac{1}{\pi} \operatorname{arc\,tan} \frac{2\pi}{\varphi}$; $0 < \varphi \leqslant \frac{1}{2}$; $\alpha = \sqrt{\gamma^2 + 4\pi^2}$; Φ^+ (Φ^-) are the limit values of Φ at the upper (lower) edge of the segment $[y^-, y^+]$. The solution for problem (2.2) is constructed in accordance with familiar methods [3]. As a result we will have

$$\Phi(z) = \frac{1}{Q(z)}\left(C + \frac{1}{\alpha} I(z)\right), \quad \Phi^{\pm}(y) = \frac{1}{Q^{\pm}(y)}\left(C + \frac{1}{\alpha} I^{\pm}(y)\right). \tag{2.3}$$

Here $I(z) = \frac{1}{2\pi i} \int\limits_{y^-}^{y^+} \frac{|Q(\eta)|\, w_y}{\eta - z}\, d\eta$; $Q(z) = (z - y^+)^{1-\varphi}(z - y^-)^{\varphi}$; $Q^{\pm}(y) = -e^{\mp\pi\varphi i}|Q(y)|$; $|Q(y)| = (y^+ - y)^{1-\varphi} \times (y - y^-)^{\varphi}$. Using the second formula in (2.2) and the Sokhotskii formulas for $I^{\pm}(y)$, we obtain

$$p(x, y) = -\frac{2Ci \sin \pi\varphi}{|Q(y)|} - \frac{\sin \pi\varphi}{\pi\alpha |Q(y)|} \int_{y^-}^{y^+} \frac{|Q(\eta)| w_y}{\eta - y} d\eta - \frac{\cos \pi\varphi}{\alpha} w_y. \quad (2.4)$$

The quantity C contained in (2.3) and (2.4) is determined from the known load q in the section. Indeed,

$$q = \int_{y^-}^{y^+} p(x, y)\, dy = \int_{y^-}^{y^+} (\Phi^+(y) - \Phi^-(y))\, dy = 2\pi i \operatorname*{Res}_{z=\infty} [\Phi(z)].$$

The residue at infinity from the term containing I(z) is equal to zero, while Q(z) ~ z as $z \to \infty$, so that $\operatorname*{Res}_{z=\infty}[\Phi(z)] = -2\pi i C$. Thus, $C = -q/(2\pi i)$.

The right-hand side in Eq. (1.2) does not depend on y. We will denote it in terms of A(x) and introduce the function

$$F(x, y) = -\int_{y}^{y^+(x)} p(x, \eta)\, d\eta.$$

Subtracting by parts the first integral in the right-hand side of (1.2), we have

$$\int_{y^-(x)}^{y^+(x)} \frac{F(x, \eta)}{\eta - y} d\eta - \frac{\gamma}{2} F(x, y) = \frac{U(x, y)}{2},$$

$$U(x, y) = w(x, y) - A(x) - 2q(x) \ln [y - y^-(x)].$$

Solution of this equation is provided by a formula analogous to (2.4), with substitution of $w_y(x, y)$ by U(x, y). However, unlike Eq. (2.1), here it is necessary that we require the boundedness of F(x, y) on the segment $[y^-(x), y^+(x)]$, since there must exist q(x). Since

$$F(x, y) = -\frac{\sin \pi\varphi}{\pi\alpha |Q(y)|} \left[\int_{y^-}^{y^+} \frac{|Q(\eta)| U(x, \eta)}{\eta - y} d\eta + C' \right] - \frac{\cos \pi\varphi}{\alpha} U(x, y) \quad (2.5)$$

(C' is an arbitrary constant) bounded when $y = y^-$, so that the expression in the brackets for $y = y^-$ must be equal to zero. Thus,

$$C' = -\int_{y^-}^{y^+} \left(\frac{y^+ - \eta}{\eta - y^-} \right)^{1-\varphi} U(x, \eta)\, d\eta.$$

After we have substituted this value into (2.5), we find that the expression in brackets is equal to

$$(y - y^-) \int_{y^-}^{y^+} \left(\frac{y^+ - \eta}{\eta - y^-} \right)^{1-\varphi} \frac{U(x, \eta)}{\eta - y} d\eta.$$

It has to be equated to zero when $y = y^+$ owing to the boundedness of F(x, y). Hence,

$$\int_{y^-}^{y^+} \frac{U(x, \eta)\, d\eta}{(y^+ - \eta)^{\varphi} (\eta - y^-)^{1-\varphi}} = 0. \quad (2.6)$$

Using the tabulated integral from [4]

$$\int_{y^-}^{y^+} (y-y^-)^{\varphi-1}(y^+-y)^{-\varphi}\,dy = \frac{\pi}{\sin\pi\varphi}$$

and the expression for U(x, y), from (2.6) we have

$$A(x) = \frac{\sin\pi\varphi}{\pi}\left[\int_{y^-(x)}^{y^+(x)} \frac{w(x,y)\,dy}{(y^+(x)-y)^{\varphi}(y-y^-(x))^{1-\varphi}} + \right.$$
$$\left. + 2q(x)\int_{y^-(x)}^{y^+(x)} \frac{\ln(y-y^-(x))}{(y^+(x)-y)^{\varphi}(y-y^-(x))^{1-\varphi}}\,dy\right].$$

The second term in the brackets, as a result of the substitution $y = y^-(x) + [y^+(x) - y^-(x)]t$, assumes the form

$$2q(x)\int_0^1 \frac{\ln t\,dt}{(1-t)^{\varphi}t^{1-\varphi}} + \frac{2q(x)\pi}{\sin\pi\varphi}\ln[y^+(x)-y^-(x)].$$

Here the integral is equal to [4, p. 502] $\frac{\pi}{\sin\pi\varphi}[\psi(\varphi)-\psi(1)]$, where $\psi(x) = -C + (x-1)\sum_{k=0}^{\infty} \times \frac{1}{(k+1)(k+x)}$, C is the Euler constant. Using the derived expression for A(x), we can write Eq. (1.2) in final form:

$$\int_{x^-}^{x^+} \frac{q(\xi)-q(x)}{|\xi-x|}\,d\xi + q(x)\left\{\ln\left[\frac{4(x^+-x)(x-x^-)}{\varepsilon^2[y^+(x)-y^-(x)]^2\sin^2\beta}\right]-\right.$$
$$\left.-2\psi(\varphi)+2\psi(1)\right\} = \frac{\sin\pi\varphi}{\pi}\int_{y^-(x)}^{y^+(x)} \frac{w(x,y)\,dy}{(y^+(x)-y)^{\varphi}(y-y^-(x))^{1-\varphi}}. \quad (2.7)$$

Since the intensity of wear is usually not high, γ is small and, consequently, φ is close to 1/2. Therefore, $\psi(\varphi) \approx \psi(1/2) + \psi'(1/2)(\varphi - 1/2)$. Using the formulas [4, p. 774] $\psi(1/2 + z) - \psi(1/2 - z) = \pi\tan\pi z$, $\psi(1/2) = -C - 2\ln 2$, we have $\psi'(1/2) = \pi^2/2$, $\psi(\varphi)-\psi(1) = -\ln 4 + (\pi^2/2)(\varphi - 1/2) \approx -\ln 4 - \gamma/4$. Let us now turn to finding solutions for Eq. (2.7).

3. Let $\varepsilon \to 0$. Then from (2.7), in the main, it follows that

$$q(x) = \frac{\sin\pi\varphi}{\pi\ln\frac{1}{\varepsilon^2}}\int_{y^-(x)}^{y^+(x)} \frac{w(x,y)\,dy}{(y-y^-(x))^{1-\varphi}(y^+(x)-y)^{\varphi}}.$$

Thus, given highly elongated contact the load in the section is entirely defined by the elastic displacement within that same section (the method of independent plane sections). If ε is not excessively small, the quantity $\ln 1/\varepsilon^2$ is not small and all of the terms in Eq. (2.7) should be taken into consideration. We will construct its exact solutions.

Let us assume that $x^{\pm} = \pm 1$, $d(x) = (1/2)(y^+(x) - y^-(x)) = \sqrt{1 - x^2}$ (in this case L and B are equal to half the length and width of contact). The half width of contact depends on x in accordance with the law for an ellipse, although the contact need not be elliptical. We will denote as follows: $y_0(x) = (1/2)[y^+(x) + y^-(x)]$, $g(x, y) = w(x, y + y_0(x))$. Equation (2.7) assumes the form

$$\int_{-1}^{1} \frac{q(\xi)-q(x)}{|\xi-x|}\,d\xi + K'q(x) = \frac{\sin\pi\varphi}{\pi}\int_{-d(x)}^{d(x)} \frac{g(x,y)\,dy}{(y+d(x))^{1-\varphi}(d(x)-y)^{\varphi}}, \quad (3.1)$$

$$K' = \ln \frac{1}{\varepsilon^2 \sin^2 \beta} - 2\psi(\varphi) + 2\psi(1).$$

In the case of limited wear intensity $K' \approx \ln \frac{16}{\varepsilon^2 \sin^2 \beta} + \frac{\gamma}{2}$. We will assume that g(x, y) = $a_0(x) + a_1(x)y + a_2(x)y^2$. In the following we need values for the integrals

$$I_m(d, \varphi) = \int_{-d}^{d} \frac{x^m \, dx}{(d-x)^\varphi (d+x)^{1-\varphi}},$$

$$J_m(d, y, \varphi) = \int_{-d}^{d} \frac{(d-x)^{1-\varphi} (d+x)^\varphi}{x-y} x^m \, dx, \quad |y| < d.$$

These are found by standard TFKP methods (the calculation is not presented here):

$$I_m(d, \varphi) = \frac{\pi}{\sin \pi\varphi} \gamma_m(\varphi) \, d^m,$$

$$\gamma_m(\varphi) = \sum_{k=0}^{m} (-1)^k \frac{(-\varphi - k + 1)_k (\varphi - m + k)_{m-k}}{k! \, (m-k)!},$$

$$\gamma_0(\varphi) = 1, \ \gamma_1(\varphi) = 2\varphi - 1, \ \gamma_2(\varphi) = 2\varphi^2 - 2\varphi + 1;$$

$$J_m(d, y, \varphi) = -\pi \operatorname{ctg} \pi\varphi \, (y+d)^\varphi (d-y)^{1-\varphi} y^m - \frac{\pi}{\sin \pi\varphi} D_{m+1}(d, y, \varphi),$$

$$D_m(d, y, \varphi) = \sum_{k=0}^{m} \rho_{m-k}(\varphi) \, y^k d^{m-k},$$

$$\rho_s(\varphi) = \sum_{k=0}^{s} (-1)^{s-k} \frac{(\varphi - k + 1)_k (2 - \varphi - s + k)_{s-k}}{k! \, (s-k)!},$$

$$\rho_0(\varphi) = 1, \ \rho_1(\varphi) = 2\varphi - 1, \ \rho_2(\varphi) = 2\varphi^2 - 2\varphi, \ \rho_3(\varphi) = \\ = (2/3)\varphi(1-\varphi)(1-2\varphi).$$

In these formulas $(a)_k = a(a+1) \ldots (a+k-1)$ is the Pokhhammer symbol [$(a)_0 = 1$]. Having introduced the notation Z(x, y) = p(x, y + y_0(x)), from (2.4) we obtain

$$Z(x, y) = -\frac{\sin \pi\varphi}{\alpha\pi} (y+d)^{-\varphi} (d-y)^{\varphi-1} [a_1 J_0(d, y, \varphi) + 2a_2 J_1(d, y, \varphi)] -$$
$$-\frac{\cos \pi\varphi}{\alpha}(a_1 + 2a_2 y) + q \frac{\sin \pi\varphi}{\pi} (y+d)^{-\varphi} (d-y)^{\varphi-1} =$$
$$= (y+d)^{-\varphi} (d-y)^{\varphi-1} \left\{ \frac{\sin \pi\varphi}{\pi} q + \frac{a_1}{\alpha} [(2\varphi - 1) d + y] + \right.$$
$$\left. + \frac{2a_2}{\alpha} [y^2 + (2\varphi - 1) dy + (2\varphi^2 - 2\varphi) d^2] \right\}.$$

We will seek the bounded distribution for the pressure. With y = ±d the expression in the braces must be equal to zero. Consequently, ($2/\alpha = \sin \pi\varphi / \pi$),

$$2\varphi d a_1 + 4d^2 \varphi^2 a_2 = -2q, \ 2(\varphi - 1) d a_1 + 4d^2(\varphi - 1)^2 a_2 = -2q.$$

After we have solved this system, we find that

$$a_1 = 2(1 - 2\varphi) a_2 d, \ q = 2\varphi(\varphi - 1) a_2 d^2;$$

$$Z(x, y) = -\frac{2a_2(x)}{\alpha} (y + d(x))^{1-\varphi} (d(x) - y)^\varphi, \quad p(x, y) = \\ = -\frac{2a_2(x)}{\alpha} (y - y^-(x))^{1-\varphi} (y^+(x) - y)^\varphi. \tag{3.2}$$

We will calculate the right-hand side of Eq. (3.1):

$$\frac{\sin \pi\varphi}{\pi} \int_{-d}^{d} \frac{g(x, y) \, dy}{(y+d)^{1-\varphi} (d-y)^\varphi} = \frac{\sin \pi\varphi}{\pi} [a_0 I_0(d, \varphi) + a_1 I_1(d, \varphi) + a_2 I_2(d, \varphi)] =$$

$$= a_0 + (2\varphi - 1) a_1 d + (2\varphi^2 - 2\varphi + 1) a_2 d^2 = a_0 + \frac{6\varphi^2 - 6\varphi + 1}{2\varphi(1-\varphi)} q.$$

Equation (3.1) assumes the form

$$\int_{-1}^{1} \frac{q(\xi) - q(x)}{|\xi - x|} d\xi + Kq(x) = a_0(x), \quad K = K' - \frac{6\varphi^2 - 6\varphi + 1}{2\varphi(1-\varphi)}. \tag{3.3}$$

The derived equation was examined in [2, 5], where its class of polynomial solutions was found.

Let $a_0(x) = \ell_0 + \ell_1 x + \ell_2 x^2$. Assuming that $q(x) = \ell_0' + \ell_1' x + \ell_2' x^2$, substituting into (3.3), and calculating the integral, we come to equation $-3\ell_2' x^2 + \ell_2' - 2\ell_1' x + K(\ell_0' + \ell_1' x + \ell_2' x^2) = \ell_0 + \ell_1 x + \ell_2 x^2$, from which it follows that

$$l_2' = \frac{l_2}{K-3}, \quad l_1' = \frac{l_1}{K-2}, \quad l_0' = \frac{1}{K}\left(l_0 - \frac{l_2}{K-3}\right). \tag{3.4}$$

4. Using the results from Sec. 3, we will solve the problem of the contact between two paraboloids. Let the equations

$$z_1 = \frac{x_1'^2}{2R_x^1} + \frac{y_1'^2}{2R_y^1}, \quad z_1 = \frac{x_1'^2}{2R_x^2} + \frac{y_1'^2}{2R_y^2} \left(\frac{1}{R_x^1} > \frac{1}{R_x^2}, \frac{1}{R_y^1} > \frac{1}{R_y^2}\right)$$

specify the surface of the first and second bodies in the rectangular system of coordinates $Ox_1'y_1'z_1$ (see Fig. 1). For the case in which R_x^2, $R_y^2 \to \infty$ the second body is a half space. Given the corresponding selection of the quantities $R_x^{1,2}$, $R_y^{1,2}$, the cited equations describe the shapes of the sphere and ring surfaces within the bearing. Subsequently,

$$w_1 = \Delta - \frac{x_1'^2}{2R_x} - \frac{y_1'^2}{2R_y}, \quad \frac{1}{R_x} = \frac{1}{R_x^1} - \frac{1}{R_x^2}, \quad \frac{1}{R_y} = \frac{1}{R_y^1} - \frac{1}{R_y^2},$$

Δ is the total elastic displacement at the coordinate origin. We will assume that $R_x \gg R_y$. In this case, the contact will be elongated. Since $x_1' = x_1 + y_1 \cos\beta$, $y_1' = y_1/\sin\beta$, then

$$w_1(x_1, y_1) = \Delta - \frac{x_1^2}{2R_x} - \frac{\cos\beta}{R_x} x_1 y_1 - \frac{y_1^2}{2R_\beta}, \quad R_\beta = \left(\frac{\cos^2\beta}{R_x} + \frac{\sin^2\beta}{R_y}\right)^{-1}.$$

In dimenisonless variables

$$w(x, y) = \left(\frac{\Delta}{B} - \frac{L^2}{2BR_x} x^2\right) - \frac{L\cos\beta}{R_x} xy - \frac{B}{2R_\beta} y^2.$$

Consequently,

$$a_0(x) = w(x, y_0(x)),$$
$$a_1(x) = \frac{\partial}{\partial y} w(x, y_0(x)) = -\frac{L\cos\beta}{R_x} x - \frac{B}{R_\beta} y_0(x), \quad a_2(x) = -\frac{B}{2R_\beta}. \tag{4.1}$$

Since the distribution of pressure is bounded, we have $q(-1) = q(1) = 0$. Then $\ell_1' = \ell_1 = 0$. It follows from the second of the equations in (3.2) that

$$q(x) = l_0' + l_2' x^2 = \varphi(1-\varphi)\frac{B}{R_\beta}(1 - x^2).$$

Thus, $\ell_0' = -\ell_2' = \varphi(1 - \varphi)B/R_\beta$. From (3.4) we find that

$$l_2 = \frac{(3-K)\varphi(1-\varphi)B}{R_\beta}, \quad l_0' = \frac{(K-1)\varphi(1-\varphi)B}{R_\beta}. \tag{4.2}$$

Using the first of the equations in (4.1) and the fact that $a_0(x) = \ell_0 + \ell_2 x^2$, we derive the equation

$$\frac{B}{2R_\beta} y_0^2(x) + \frac{L\cos\beta}{R_x} xy_0(x) + l_0 - \frac{\Delta}{B} + \left(\frac{L^2}{2BR_x} + l_2\right) x^2 = 0,$$

from which it follows that

$$y_0(x) = -\frac{LR_\beta \cos\beta}{BR_x} x \pm \sqrt{F_1 + F_2 x^2},$$
$$F_1 = 2\frac{R_\beta}{B}\left(\frac{\Delta}{B} - l_0\right), \quad F_2 = \frac{L^2 R_\beta^2 \cos^2\beta}{B^2 R_x^2} - 2\frac{R_\beta}{B}\left(l_2 + \frac{L^2}{2BR_x}\right). \tag{4.3}$$

Using the first relationship in (3.2) and the second relationship in (4.1), we have

$$\mp \frac{B}{R_\beta}\sqrt{F_1 + F_2 x^2} = \frac{(2\varphi - 1)B}{R_\beta}\sqrt{1 - x^2}. \tag{4.4}$$

Since $2\varphi - 1 < 0$, then in (4.3) and in (4.4) we have to choose the upper sign. Subsequently, as we can see from (4.4), $F_1 = -F_2 = (1 - 2\varphi)^2$. Using expression (4.3) for F_1 and F_2 and expression (4.2) for ℓ_0 and ℓ_2, after transformation we obtain

$$B = \left(\frac{2R_\beta\Delta}{k_0}\right)^{1/2}, \quad \frac{R_\beta}{R_x} = \frac{1 - \sqrt{1 - 4k_1\varepsilon^2\cos^2\beta}}{2\cos^2\beta}, \tag{4.5}$$

where $k_0 = (1 - 2\varphi)^2 + 2(K - 1)\varphi(1 - \varphi)$, $k_1 = (1 - 2\varphi)^2 + 2(K - 3)\varphi(1 - \varphi)$. Using the determination of the quantity R_β, we find

$$\frac{R_y}{R_x} = \mathrm{tg}^2\beta\,\frac{1 - \sqrt{1 - 4k_1\varepsilon^2\cos^2\beta}}{1 + \sqrt{1 - 4k_1\varepsilon^2\cos^2\beta}}. \tag{4.6}$$

Let us draw a number of conclusions. The half width of the contact depends on x in accordance with the law for an ellipse. The mean line and the "boundaries" of contact are given by the formulas

$$y_0(x) = -\frac{R_\beta\cos\beta}{\varepsilon R_x} x + (1 - 2\varphi)\sqrt{1 - x^2}, \quad y^+(x) =$$
$$= -\frac{R_\beta\cos\beta}{\varepsilon R_x} x + (2 - 2\varphi)\sqrt{1 - x^2}, \quad y^-(x) = -\frac{R_\beta\cos\beta}{\varepsilon R_x} x - 2\varphi\sqrt{1 - x^2}$$

and represent the arcs of ellipses contained between tangents parallel to the velocity of slippage, while the first term in these formulas defines the straight line connecting the points of tangency. The quantity ε is found from the transcendental equation (4.6). From formula (4.5), with Δ given, we derive the maximum half width of contact B and $L = B/\varepsilon$.

Let us determine the normal force P, the components of the tangential force T_x, T_y, and the components of the moment M_x, M_y (on the Ox_1', Oy_1' axes), acting on body 1 and governed by the distribution of pressure, denoting in terms of G_1' the area of contact in coordinates x_1', y_1':

$$P = \iint\limits_{G_1'} p_1\,dx_1'\,dy_1' = \sin\beta \iint\limits_{G_1'} p_1\,dx_1\,dy_1 = \frac{LB}{\theta}\int_{-1}^{1} q(x)\,dx = \frac{4\varphi(1 - \varphi)LB^2}{3\theta R_\beta}.$$

Then

$$L = \left(\frac{3\theta R_\beta P}{4\varphi(1 - \varphi)\varepsilon^2}\right)^{1/3}, \quad B = \left(\frac{3\varepsilon\theta R_\beta P}{4\varphi(1 - \varphi)}\right)^{1/3}.$$

Let us find T_x and T_y. Let the equation $z_1 = \chi(x_1', y_1')$ specify the surface of the deformed body 1. With accuracy to an inconsequential additive constant

$$\chi(x_1', y_1') = \frac{x_1'^2}{2R_x^1} + \frac{y_1'^2}{2R_y^1} + \theta_1 S(x_1', y_1'), \quad S(x_1', y_1') =$$

$$= \iint_{G_1'} \frac{p_1(\xi_1', \eta_1')\, d\xi_1'\, d\eta_1'}{\sqrt{(x_1' - \xi_1')^2 + (y_1' - \eta_1')^2}}.$$

Projections onto the Ox_1', Oy_1' axes of the unit normal to the surface are defined by the formulas

$$n_x = -\frac{\partial \chi}{\partial x_1'}, \quad n_y = -\frac{\partial \chi}{\partial y_1'},$$

while the components of the tangential force

$$T_x = -\iint_{G_1'} p_1 \frac{\partial \chi}{\partial x_1'}\, dx_1'\, dy_1', \quad T_y = -\iint_{G_1'} p_1 \frac{\partial \chi}{\partial y_1'}\, dx_1'\, dy_1'.$$

These expressions will include the following terms:

$$A_x = \iint_{G_1'} p_1 \frac{\partial S}{\partial x_1'}\, dx_1'\, dy_1', \quad A_y = \iint_{G_1'} p_1 \frac{\partial S}{\partial y_1'}\, dx_1'\, dy_1'.$$

We will demonstrate that they are equal to zero. Indeed,

$$\iint_{G_1'} p_1 \frac{\partial S}{\partial x_1'}\, dx_1'\, dy_1' = -\iint_{G_1'} p_1(x_1', y_1') \left[\iint_{G_1'} \frac{p_1(\xi_1', \eta_1')(x_1' - \xi_1')\, d\xi_1'\, d\eta_1'}{[(x_1' - \xi_1')^2 + (y_1' - \eta_1')^2]^{3/2}}\right] dx_1'\, dy_1'.$$

Changing the order of integration over x_1', y_1' and ξ_1', η_1', we obtain an expression different from the right-hand side of the last equation only in terms of sign. Consequently, $A_x = 0$. Analogously, $A_y = 0$. Let us note that the proved equations are, essentially, a consequence of Newton's third law (tangential forces acting on a body are equal in absolute value and oppositely directed). Using that which has been proved, we obtain

$$T_x = -\iint_{G_1'} \frac{x_1'}{R_x^1} p_1(x_1', y_1')\, dx_1'\, dy_1' = -\frac{L^2 B}{\theta R_x^1} \iint_G (x + \varepsilon y \cos\beta)\, p\, dx\, dy =$$

$$= -\frac{LB^2}{\theta R_x^1} \cos\beta \int_{-1}^{1} dx \int_{y^-(x)}^{y^+(x)} yp\, dy.$$

The internal integral

$$\int_{-d(x)}^{d(x)} Z(x, y)(y + y_0(x))\, dy = y_0(x)\, q(x) - \frac{2a_2(x)}{\alpha} J_2(d(x), 0, 1 - \varphi) =$$

$$= y_0(x)\, q(x) + \frac{B}{R_\beta}(1 - x^2)^{3/2} \frac{\varphi(1 - \varphi)(1 - 2\varphi)}{3}.$$

Utilizing the equation

$$\int_{-1}^{1} y_0(x)\, q(x)\, dx = \frac{\varphi(1 - \varphi) B}{R_\beta} \int_{-1}^{1} (1 - x^2) \left[-\frac{LR_\beta \cos\beta}{BR_x} x + \right.$$

$$\left. + (1 - 2\varphi)\sqrt{1 - x^2}\right] dx = \frac{3\pi\varphi(1 - \varphi)(1 - 2\varphi) B}{8R_\beta},$$

we obtain $T_x = -\frac{LB^3}{\theta R_x^1}\cos\beta\frac{\pi\varphi(1-\varphi)(1-2\varphi)}{2R_\beta}$, and this is followed by $T_y = -\iint_{G_1'}\frac{y_1'}{R_y^1}p_1(x_1', y_1')\,dx_1'\,dy_1' =$ $-\frac{LB^2}{\theta R_y^1}\sin\beta\int_{-1}^{1}dx,$

$$\int_{y^-(x)}^{y^+(x)} py\,dy = -\frac{LB^3}{\theta R_y^1}\sin\beta\frac{\pi\varphi(1-\varphi)(1-2\varphi)}{2R_\beta}.$$

The orthogonal projection of the tangential force onto the vector $\mathbf{V}$ (the force of resistance)

$$T_r = -T_x\cos\beta - T_y\sin\beta = \pi\varphi(1-\varphi)(1-2\varphi)\frac{LB^3}{2\theta R_\beta}\left(\frac{\cos^2\beta}{R_x^1}+\frac{\sin^2\beta}{R_y^1}\right),$$

while the projection onto the vector $[\mathbf{V}, \mathbf{e}_z]$ (the side force)

$$T_l = \pi\varphi(1-\varphi)(1-2\varphi)\sin\beta\cos\beta\frac{LB^3}{2\theta R_\beta}\left(\frac{1}{R_x^1}-\frac{1}{R_y^1}\right). \tag{4.7}$$

As we can see from (4.7), the side force acting on body 1, when $1/R_y^1 > 1/R_x^1$ ($1/R_y^1 < 1/R_x^1$) is directed in that direction away from the vector $\mathbf{V}$, in which the angle between $\mathbf{V}$ and Ox_1 is acute (obtuse). When $\beta = \pi/2$ or $R_x^1 = R_y^1$ (this equality is satisfied, for example, in the contact of a sphere with the ring of the bearing) the side force is equal to zero. If there is no wear ($\varphi = 1/2$), so that $T_r = T_\ell = 0$.

Let us now turn to the calculation of the moments:

$$M_x = \iint_{G_1'} p_1 y_1'\,dx_1'\,dy_1' = \frac{\sin\beta}{\theta}\iint_G py_1\,dx_1\,dy_1 = \frac{LB^2\sin\beta}{\theta}\int_{-1}^{1}dx\int_{y^-(x)}^{y^+(x)} py\,dy.$$

The internal integral has already been calculated in the determination of T_x, so that

$$M_x = \frac{\pi LB^3}{2\theta R_\beta}\sin\beta\varphi(1-\varphi)(1-2\varphi) = -T_y R_y^1. \tag{4.8}$$

Further,

$$M_y = -\iint_{G_1'} p_1 x_1'\,dx_1'\,dy_1' = -\frac{1}{\theta}\iint_{G_1} p_1(x_1 + y_1\cos\beta)\,dx_1\,dy_1 =$$

$$= -\frac{L^2B}{\theta}\left[\iint_G px\,dx\,dy + \frac{B}{L}\cos\beta\iint_G py\,dx\,dy\right].$$

The first term in the brackets is equal to zero and, therefore,

$$M_y = -M_x\operatorname{ctg}\beta = T_x R_x^1. \tag{4.9}$$

As we can see from formulas (4.8) and (4.9), the moment is orthogonal to the sliding velocity and leads to the reversal of body 1, while its modulus

$$M = \sqrt{M_x^2 + M_y^2} = \frac{LB^3}{2\theta R_\beta}\varphi(1-\varphi)(1-2\varphi).$$

If there is no wear, the moment is equal to zero. The center of pressure is found on the positive Oy_1 half axis at a distance $3B(1 - 2\varphi)/8$ from point O.

The problem examined at this point generalized the Hertz problem to the case of narrow contact in the presence of wear.

LITERATURE CITED

1. L. A. Galin, The Contact Problem in the Theory of Elasticity and Viscoelasticity [in Russian], Nauka, Moscow (1980).

2. A. N. Burmistrov, "The contact problem in the theory of elasticity for elongated dyes," in: An Investigation into the Nonsteady Motion of a Continuous Medium: An Interdisciplinary Collection [in Russian], MFTI, Moscow (1987).
3. F. D. Gakhov, Boundary-Value Problems [in Russian], Nauka, Moscow (1977).
4. A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev, Integrals and Series. Elementary Functions [in Russian], Nauka, Moscow (1981).
5. A. N. Burmistrov, "The pressure of an elongated dye on an elastic half space," Trenie Iznos, 9, No. 3 (1988).

WAVEGUIDE EFFECT IN A ONE-DIMENSIONAL PERIODICALLY PENETRABLE STRUCTURE

S. V. Sukhinin

UDC 517.947+534.2+535.42

The waveguide properties of permeable one-dimensional periodic acoustic structures are studied here. These waveguide properties are associated with the existence of intrinsic waves localized in the vicinity of the structure. Their properties are described by generalized eigenfunctions which are solutions of problems describing the steady oscillations about the structure. The possibility of the existence of generalized eigenfunctions localized in the vicinity of a one-dimensional periodic penetrable layer or about a periodic chain of permeable barriers is demonstrated in this study. Examples are presented of the waveguide permeable periodic structures for which the boundaries asymptotic with respect to limited permeability or with respect to special geometric shape are studied, and also the properties of the natural oscillations, and eigenvalues are determined. These examples may serve as models both for experimental and numerical studies into the waveguide properties of a periodically permeable structure.

1. Formulation of the Problems and Necessary Information. Let a space be filled with a medium in which the speed of sound is represented by c_2 and the density in a state of rest is represented by ρ_2. The medium contains either a one-dimensional periodic layer (Fig. 1a) or a string of inclusions (Fig. 1b) of another medium, where the speed of sound is c_1, and the density in a state of rest is ρ_1. It is assumed that the boundary between these media is periodic along the y axis, with a period of 2π. It is assumed, further, that all motion within the media depends exclusively on two spatial variables: x, y. It is therefore convenient to utilize the following notation: Ω_1 is the area on the (x, y) plane which simulates the layer or chain of inclusions, while Ω_2 models the area filled with the external medium, and Γ represents the boundary between these media (see Fig. 1).

Let $f(x, y)\exp(-i\omega t)$ describe the periodic sources of the sound. It is assumed that the sources are situated in the medium Ω_2, positioned periodically along the y axis with a period 2π, ω is the angular frequency of the oscillations. The sound waves are described

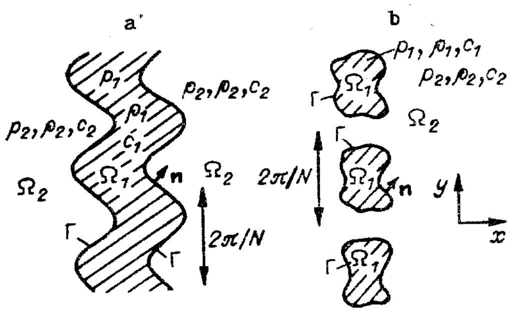

Fig. 1

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